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ONE APPROXIMATE SOLUTION OF THE NEKRASOV PROBLEM

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An approximate solution $\omega = A[\omega, \mu]$ of the nonlinear integral Nekrasov equation is obtained by successive replacement of the kernel of the integral operator by a close one. The solution is sought not directly at the bifurcation point $\mu_1 = 3$ of the linearized equation $\omega = \mu L[\omega]$ but at the point $\mu = 1$ at which operator $A[\omega, \mu]$, remaining nonlinear in ω, is linear in μ.

Key words: *integral equation, nonlinear operator, iterative method, motionless point.*

1. Formulation of the Problem. The well-known Nekrasov problem of free nonlinear waves on the surface of a fluid of infinite depth [1] is given by the system of differential equations

$$\frac{dx}{d\theta} = -\frac{\lambda}{2\pi} R(\theta)\cos\omega(\theta); \tag{1.1}$$

$$\frac{dy}{d\theta} = -\frac{\lambda}{2\pi} R(\theta)\,\sin\omega(\theta); \tag{1.2}$$

$$\frac{d}{d\theta}\,\ln R(\theta) = -\frac{1}{3}\,\mu\sin\omega(\theta)\Big(1 + \int\limits_0^\theta \sin\omega(\sigma)\,d\sigma\Big)^{-1}. \tag{1.3}$$

Here λ is the wavelength and μ is a parameter that depends on the wavelength λ, the wave velocity c, and the acceleration due to gravity g; the x axis is directed horizontally, the y axis is directed vertically upward, and the coordinate origin O is at the wave crest (Fig. 1); the odd function $\omega(\theta)$, which vanishes at the point $\theta = 0$, satisfies the Nekrasov integral equation in the theory of steady-state nonlinear periodic waves:

$$\omega(\theta) = \int\limits_0^{2\pi} K(\theta,\sigma)\mu\sin\omega(\sigma)\Big(1 + \mu\int\limits_0^\sigma \sin\omega(s)\,ds\Big)^{-1} d\sigma. \tag{1.4}$$

Here the kernel

$$K(\theta,\sigma) = \frac{1}{3}\sum_{n=1}^{\infty}\frac{\sin n\theta\sin n\sigma}{n\pi}$$

is Green's function of the Neumann problem for the Laplace operator in a unit circle. The expression on the right side of Eq. (1.4) is denoted by $A[\omega, \mu]$ and treated as a nonlinear operator which acts in the space of 2π-periodic functions in the interval $[0, 2\pi]$. The existence of a nontrivial solution of the integral equation

$$\omega(\theta) = A[\omega, \mu] \tag{1.5}$$

was proved in [1] and [2, 3]. It has been shown [1–3] that the operator $A[\omega, \mu]$ is continuous and compact in a small

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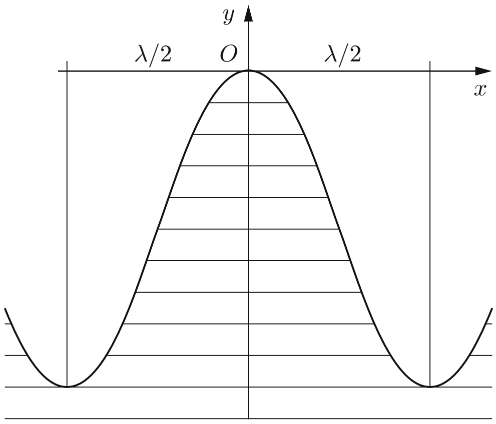

Fig. 1. Wave profile.

sphere in a small vicinity of zero. Performing Frechet differentiation of the right side of Eq. (1.5) at zero, we obtain the linearized equation

$$\omega(\theta) = \mu L[\omega], \tag{1.6}$$

where

$$L[\omega] = \int_0^{2\pi} K(\theta, \sigma)\omega(\sigma)\, d\sigma.$$

The spectrum of Eq. (1.6) consists of simple eigenvalues $\mu_k = 3k$ ($k = 1, 2, \ldots$) to which the eigenfunctions $e_k = \sin k\theta$ correspond. The values μ_k are the bifurcation points of Eqs. (1.5) at which waves occur. Actually, only the first bifurcation point [4] has a physical meaning.

Solution of the operator equation (1.5) involves great difficulties due to the complex dependence of the operator $A[\omega, \mu]$ on the parameter μ. In [5, 6], it was proved that the representation

$$A[\omega, \mu] = \mu L[\omega] + T[\omega, \mu]$$

($T[\omega, \mu]$ is a continuous nonlinear operator) is valid and that approximate solutions of the operator equations (1.5) can be constructed in the vicinity of the bifurcation points using the Lyapunov–Schmidt method. Direct application of the Lyapunov–Schmidt method to the solution of Eq. (1.5) leads to the expression [1, 5]

$$\omega(\theta, \varepsilon) = \Big(\frac{1}{9}\,\varepsilon - \frac{8}{243}\,\varepsilon^2 + \frac{115}{13\,122}\,\varepsilon^3 + \ldots\Big) \sin\theta + \Big(\frac{1}{54}\,\varepsilon^2 - \frac{8}{729}\,\varepsilon^3 + \ldots\Big) \sin 2\theta + \frac{17}{4374}\,\varepsilon^3 \sin 3\theta + O(\varepsilon^4). \tag{1.7}$$

The parameter ε, which characterizes the wave amplitude is determined from the relation $\mu = \mu_1 + \varepsilon$, where μ is a solution of the nonlinear equation (1.5) in the vicinity of the point μ_1. In the present paper, the nonlinear integral equation (1.5) is solved using a different method, which is described in [7]. This solution is constructed only in the vicinity of the point μ_1 taking into account that solutions in the vicinities of any bifurcation points μ_n ($n \geq 1$) can be found using the same method. In this method, a calculation is made of the parameter ε, which is then used to calculate the solution $\omega(\theta, \varepsilon)$ from formula (1.7) obtained using the Lyapunov–Schmidt method.

2. Solution of the Integral Nekrasov Equation. To solve Eq. (1.5) we use the method of replacing the kernel by a close kernel [8, p. 552]. In this case, the kernel $K(\theta, \sigma)$ of Eqs. (1.5) is replaced by the truncated kernel $K_n(\theta, \sigma)$. As a result, we obtain the equation

$$\omega_n(\theta) = A_n[K_n(\theta, \sigma), \omega_n, \mu_{1n}], \tag{2.1}$$

where

$$A_n[K_n(\theta, \sigma), \omega_n, \mu_{1n}] = \int_0^{2\pi} K_n(\theta, \sigma)\mu_{1n} \sin\omega_n(\sigma)\Big(1 + \mu_{1n} \int_0^{\sigma} \sin\omega_n(s)\, ds\Big)^{-1} d\sigma,$$

$$K_n(\theta,\sigma)=\sum_{k=1}^{n}\frac{\sin k\theta\sin k\sigma}{3k\pi},$$

μ_{1n} is a solution of Eq. (1.5) for the parameter μ with the truncated kernel $K_n(\theta,\sigma)$ in the vicinity of the first bifurcation point μ_1. Convergence of the solution of Eq. (2.1) to the solution of the integral equation (1.5) follows from the convergence condition $\|K(\theta,\sigma)-K_n(\theta,\sigma)\|\to 0$ at $n\to\infty$. The rate of convergence of the sequence $\omega_k(\theta)$ $(k=1,2,\ldots)$ can be estimated by using the norm

$$\rho(\omega_n^\infty(\theta)-\omega_n(\theta))=\max_{0\le\theta\le 2\pi}|\omega_n^\infty(\theta)-\omega_n(\theta)|,$$

where

$$\omega_n^\infty(\theta)=A_n[K(\theta,\sigma),\omega_n,\mu_{1n}]. \tag{2.2}$$

For convenience of the further consideration, we use the notation

$$f_n(\sigma)=\sin\omega_n(\sigma),\qquad g_n(\sigma)=\int_0^\sigma\sin\omega_n(s)\,ds.$$

In this notation, Eq. (2.1) can be written as

$$\omega_n(\theta)=\sum_{k=1}^{n}\frac{\sin k\theta}{3k\pi}\int_0^{2\pi}\frac{\mu_{1n}\sin k\sigma f_n(\sigma)}{1+\mu_{1n}g_n(\sigma)}\,d\sigma, \tag{2.3}$$

whence it follows that its solution is the trigonometric polynomial

$$\omega_n(\theta)=\sum_{k=1}^{n}a_k\sin k\theta \tag{2.4}$$

with the coefficients a_k $(k=1,2,\ldots n)$ satisfying the integral equations

$$a_k=\int_0^{2\pi}\frac{\mu_{1n}\sin k\sigma f_n(\sigma)}{3k\pi(1+\mu_{1n}g_n(\sigma))}\,d\sigma. \tag{2.5}$$

Equation (2.1) can be written as

$$\omega_n(\theta)=\mu_{1n}B_n[\omega_n]+C[\omega_n,\mu_{1n}], \tag{2.6}$$

where

$$B_n[\omega_n]=\int_0^{2\pi}K_n(\theta,\sigma)\,\frac{f_n(\sigma)}{1+g_n(\sigma)}\,d\sigma,\quad C[\omega_n,\mu_{1n}]=\int_0^{2\pi}K_n(\theta,\sigma)\,\frac{f_n(\sigma)g_n(\sigma)\mu_{1n}(1-\mu_{1n})}{(1+g_n(\sigma))(1+\mu_{1n}g_n(\sigma))}\,d\sigma.$$

As the zero approximation for $n=1$ we use the value $\mu_{11}=1$ for which the nonlinear operator $C[\omega_n,\mu_{1n}]=0$ and Eq. (2.6) is equivalent to the equation linear in μ:

$$\omega_1^0(\theta)=\mu_{11}^0B_1[\omega_1^0(\theta)]. \tag{2.7}$$

The solution of this equation is the function $\omega_1^0=a_1^0\sin\theta$.

To calculate the parameters a_1^0 and μ_{11}^0 at which Eq. (2.7) has a nontrivial solution, we determine the sequences of the functions $\omega_{1m}^0=a_{1m}^0\sin\theta$ $(m\ge 1)$ and the numbers μ_{1m}^0 from the recursive relations

$$\omega_{1m}^0=B_1[\omega_{1m-1}^0],\qquad \mu_{11m}^0=\|\omega_{1m-1}^0\|/\|\omega_{1m}^0\|, \tag{2.8}$$

where $\|\omega_{1m}^0\|$ is the norm of the function ω_{1m}^0. The satisfaction of the Schauder motionless point principle [6], whose applicability to the operator equation Eq. (1.5) is proved in [1, 2], guarantees that the sequence ω_{1m}^0 converges to the exact solution ω_1^0 as $m\to\infty$. The second equation (2.8) can be written as $\mu_{11m}^0=\|\omega_{1m-1}^0\|(\|B_1\omega_{1m-1}^0\|)^{-1}$, whence it follows that $\mu_{11m}^0\to\mu_{11}^0=\text{const}$ as $m\to\infty$. Numerical solution of Eqs. (2.8) yields $a_1^0=0.047452$ and $\mu_{11}^0=\pi$ [7]. We fix the coefficient $a_1^0=0.047452$ and consider the value of the parameter $\mu_{11}^0=\pi$ as the first approximation of μ_{11} (above, as the zero approximation we used the value $\mu_{11}=1$).

Before solving Eqs. (2.6) for $n = 1$, we prove the following theorem [9].

Theorem 1. *Let* $\omega_1^0 = a_1^0 \sin\theta$, μ_{11}^0 *be a solution of Eq.* (2.7) *and* $a_1^0 \in [0, \pi)$, $\mu_{11}^0 \in [0, A_1[\omega_1^0, \infty] B_1^{-1}[\omega_1^0])$, *where*

$$A_1[\omega_1^0, \infty] = \int_0^{2\pi} K_1(\theta, \sigma)\, \frac{f_1(\sigma)}{g_1(\sigma)}\, d\sigma, \qquad B_1[\omega_1^0] = \int_0^{2\pi} K_1(\theta, \sigma)\, \frac{f_1(\sigma)}{1 + g_1(\sigma)}\, d\sigma.$$

Then, for $n = 1$, *Eq.* (2.1) *has a unique solution for the parameter* μ_{11} *at the point* $\omega_1 = \omega_1^0$.

Proof. The function ω_1^0 is a solution of Eq. (2.1) for $n = 1$ if

$$A_1[\omega_1^0, \mu_{11}] - \mu_{11}^0 B_1[\omega_1^0] = 0, \tag{2.9}$$

where

$$A_1[\omega_1^0, \mu_{11}] = \int_0^{2\pi} K_1(\theta, \sigma)\, \frac{\mu_{11} f_1(\sigma)}{1 + \mu_{11} g_1(\sigma)}\, d\sigma, \qquad \mu_{11}^0 B_1[\omega_1^0] = \int_0^{2\pi} K_1(\theta, \sigma)\, \frac{\mu_{11}^0 f_1(\sigma)}{1 + g_1(\sigma)}\, d\sigma.$$

In this equation, the integrands are continuous positive (by virtue of the conditions of Theorem 1) functions σ in the interval $(0, 2\pi)$. The integrand function of the integral $A_1[\omega_1^0, \mu_{11}]$ is a monotonically increasing function $\mu_{11} \in [0, \infty]$ with the maximum and minimum values of the integral, respectively:

$$A_1[\omega_1^0, 0] = 0, \qquad A_1[\omega_1^0, \infty] = \lim_{\mu_{11} \to \infty} A_1[\omega_1^0, \mu_{11}].$$

Hence, Eq. (2.9) has a unique solution if $\mu_{11}^0 \in [0, A_1[\omega_1^0, \infty] B_1^{-1}[\omega_1^0])$. Theorem 1 is proved.

The calculations yielded the inequalities $A_1[\omega_1^0, \infty] < 2.09426$ and $B_1^{-1}[\omega_1^0] < 21.07392$. Thus, the values of μ_{11}^0 that satisfy the conditions of the theorem are in the closed interval $\mu_{11}^0 \in [0; 44.13426]$. Solving Eq. (2.3) for $n = 1$ and $\omega_1 = \omega_1^0$ by using the method of consecutive approximations in the vicinity of the point $\mu_{11}^0 = \pi$, we obtain $\mu_{11} = 3.47841$. Substitution of $\omega_1(\theta)$ and μ_{11} into the right side of Eq. (2.2) yields the trigonometric series

$$\omega_1^\infty(\theta) = 0.047452 \sin\theta + 0.0016889 \sin 2\theta + 0.000081627 \sin 3\theta + \dots\ .$$

The norm $\rho(\omega_1^\infty(\theta) - \omega_1(\theta)) \simeq 0.0017$ shows that for $n = 1$, solution (2.4) is close to the exact solution. To reduce the error, we calculate $\omega_2(\theta) = a_1 \sin\theta + a_2 \sin 2\theta$. We first set $\mu_{12}^0 = \mu_{11}$ in (2.5) and calculate a_2 by solving the equation

$$a_2 = \int_0^{2\pi} \frac{\mu_{12}^0 \sin(2\sigma) f_2(\sigma)}{6\pi(1 + \mu_{12}^0 g_2(\sigma))}\, d\sigma$$

using the method of successive approximations. Then, for the known coefficients a_1, a_2, and $n = 2$, from Eq. (2.3) we find the parameter μ_{12}. As a result, we have $a_2 = 0.00336833$ and $\mu_{12} = 3.48856$. Consequently, for the truncated kernel $K_2(\theta, \sigma)$, the solution of Eq. (2.1) becomes $\omega_2 = 0.047452 \sin 2\theta + 0.00336833 \sin 2\theta$. Substitution of this solution into the right side of Eq. (2.2) yields

$$\omega_2^\infty(\theta) = 0.047452 \sin\theta + 0.00336833 \sin 2\theta + 0.000201158 \sin 3\theta + \dots\ ,$$

$$\rho(\omega_2^\infty(\theta) - \omega_2(\theta)) \simeq 0.00020.$$

Similarly, for $n = 3$, we find the values of the following approximation:

$$a_3 = 0.000302041, \qquad \mu_{13} = 3.48971;$$

$$\omega_3(\theta) = 0.047452 \sin\theta + 0.00336833 \sin 2\theta + 0.000302041 \sin 3\theta + \dots\ ; \tag{2.10}$$

$$\rho(\omega_3^\infty(\theta) - \omega_3(\theta)) \simeq 0.000022.$$

The accuracy of the solution (2.10) obtained using the approximate method is sufficiently high, and, hence, calculation of the terms of the sum on the right of Eq. (2.4) with numbers $k > 3$ is meaningless.

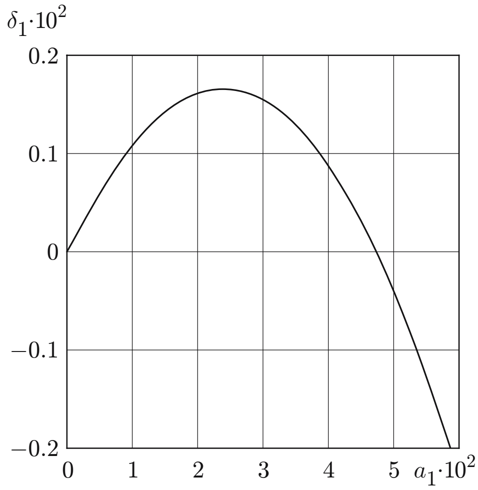

Fig. 2. Dependence $\delta_1(a_1)$.

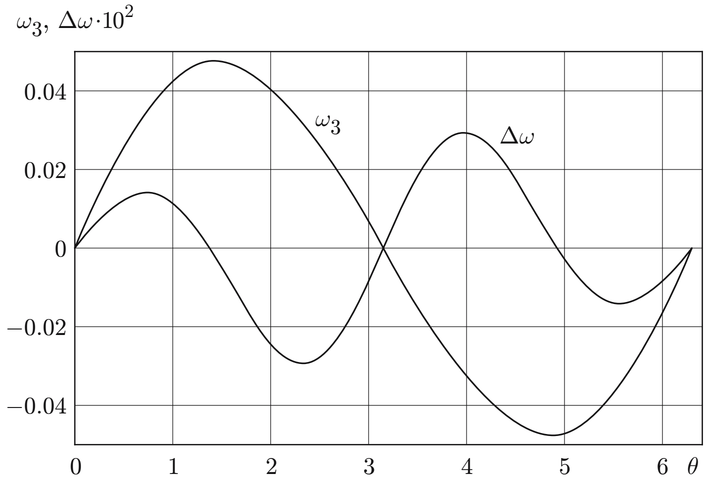

Fig. 3. Dependences $\omega_3(\theta)$ and $\Delta\omega(\theta)$.

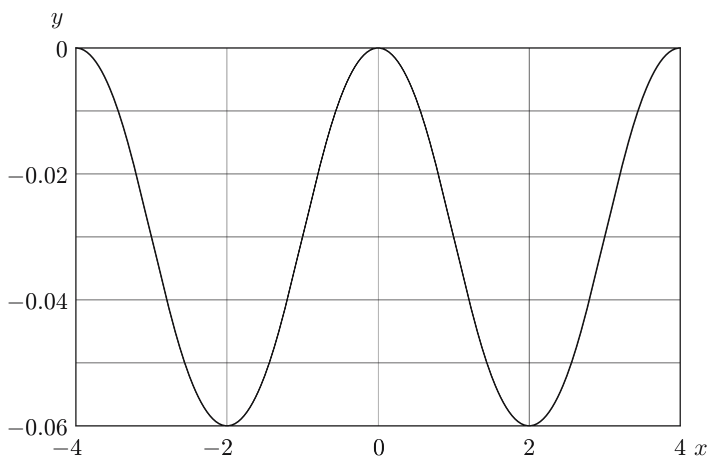

Fig. 4. Solution $y(x)$ of problem (1.1)–(1.3) for $\lambda = 1$ m and $\mu = 3.48971$.

The coefficients $a_1, a_2, \ldots$ in Eq. (2.1) are independent and determine the motionless point of Eq. (1.5). Indeed, successive scalar multiplication of the left and right sides of Eq. (2.1) into the function $\sin k\theta$ $(k = 1, 2, \ldots)$ with allowance for (2.4), (2.5) leads to the system of independent equations

$$\delta_k = a_k - \int_0^{2\pi} \frac{\mu_{1n} \sin(k\sigma) f_n(\sigma)}{3k\pi(1 + \mu_{1n} g_n(\sigma))}\, d\sigma = 0, \qquad k = 1, 2, \ldots\ ,$$

which determines the location of the motionless point in the space of the functions $\sin k\theta$ $(k = 1, 2, \ldots)$ which are orthogonal in the interval $[0, 2\pi]$. Figure 2 shows the dependence $\delta_1(a_1)$. At the motionless point $a_1 = 0.047452$, the function $\delta_1(a_1)$ vanishes.

Returning to Eq. (1.7) and setting $\mu = \mu_{13}$, we find the parameter $\varepsilon = 0.48971$. Substitution of this value into Eq. (1.7) yields

$$\omega(\theta, \varepsilon) = 0.047546 \sin\theta + 0.00315225 \sin 2\theta + 0.000456443 \sin 3\theta + O(\varepsilon^4); \tag{2.11}$$

$$\rho(\omega_3(\theta) - \omega(\theta, \varepsilon)) \simeq 0.000\,41.$$

A comparison of Eqs. (2.10) and (2.11) shows that the approximate solutions obtained by different methods almost coincide. One cannot argue that these solutions are completely independent because it was not possible to calculate the parameter ε using the Lyapunov–Schmidt method. Figure 3 shows the dependences $\omega_3(\theta)$ and $\Delta\omega = (\omega_3(\theta) - \omega(\theta, \varepsilon)) \cdot 10^2$.

3. Solution of System (1.1)–(1.3). We return to system (1.1)–(1.3), for which we assume that $\omega(\theta) = \omega_3(\theta)$ and $\mu = \mu_{13}$. Integration of Eq. (1.3) yields [3]

$$\ln R(\theta) = -\frac{1}{3} \ln \Big[\frac{3\chi^2}{2\mu}\Big(1 + \mu \int_0^{\theta} \sin\omega(\sigma)\, d\sigma\Big)\Big], \tag{3.1}$$

where $\chi^2 = g\lambda/(\pi c^2)$. Solution of system (1.1), (1.2) taking into account (3.1) yields the dependence $y = y(x)$, which defines the wave profile in parametric form. The results of the calculation for $\chi^2 = \mu$ and $\lambda = 1$ m are presented in Fig. 4. The function $\omega_3(\theta)$ is determined from formula (2.10). From the calculations it follows that a wave of length $\lambda = 1$ m has an amplitude $a \simeq 0.06$ m and moves at a velocity $c \simeq 0.945$ km/sec.

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REFERENCES

1. A. I. Nekrasov, *Collected Papers*, Vol. 1: *Exact Theory of Steady-State Waves on a Heavy Fluid Surface* [in Russian], Fizmatgiz, Moscow (1961).
2. L. N. Sretenskii, *Theory of Fluid Wave Motion* [in Russian], Nauka, Moscow (1977).
3. S. A. Gabov, *Introduction to the Theory of Nonlinear Waves* [in Russian], Izd. Mosk. Gos. Univ., Moscow (1988).
4. V. C. L. Hutson and J. S. Pym, *Applications of Functional Analysis and Operator Theory*, Academic Press, New York (1980).
5. M. M. Vainberg and V. A. Trenogin, *Theory of Branching of Solutions of Nonlinear Equations* [in Russian], Nauka, Moscow (1969).
6. S. G. Krein (ed.), *Functional Analysis* [in Russian], Nauka, Moscow (1972).
7. T. A. Bodnar', "Nonlinear waves on the surface of an incompressible heavy fluid," in: *Theory and Applications of Free-Boundary Problems*, Abstracts of All-Russian. Conf. (Biisk, July 2–6, 2002), Altai Gos. Univ., Barnaul (2002), pp. 20–22.
8. L. V. Kantorovich and G. P. Akilov, *Functional Analysis* [in Russian], Nauka, Moscow (1984).
9. T. A. Bodnar', "Approximate solution of the Nekrasov equation," in: *Free-Boundary Problems: Theory, Experiment, and Applications,* Abstracts of All-Russian Conf. with the Participation of Foreign Scientists (Biisk, July 4–8, 2005), Inst. of Hydrodynamics, Sib. Div., Russian Academy of Sciences, Novosibirsk (2005), pp. 21–22.